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FINITE-AMPLITUDE ACOUSTIC PULSES IN A WAVEGUIDE LAYER WITH LONGITUDINAL INHOMOGENEITY

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This paper considers the propagation of a weakly nonlinear acoustic pulse in a slightly curved waveguide layer which is strongly inhomogeneous in the transverse direction and weakly inhomogeneous in the longitudinal direction. The basic system of hydrodynamic equations reduces to a nonlinear wave equation, whose coefficients are determined using the equations of state of the medium. It is established that as the adiabatic exponent passes through the value $\gamma = 3/2$, the nature of the pulse propagation changes: for large values of γ, the medium is focusing, and for smaller values, it is defocusing. It is shown that the pulse propagation process is characterized by three scales: the high-frequency filling is modulated by the envelope, whose evolution, in turn, is determined by the moderate-rate evolution of the envelope phase and slow amplitude variation. A generalized nonlinear Schrödinger equation with the coefficients dependent on the longitudinal coordinate is derived for the pulse envelope. An explicit soliton solution of this equation is constructed for some types of longitudinal inhomogeneity.

Key words: *acoustic pulse, nonlinear wave equation, adiabatic exponent, nonlinear Schrödinger equation with variable coefficients, envelope soliton.*

Introduction. Problems of nonlinear acoustics are solved using methods and results of the theory of nonlinear waves [1, 2]. Wave processes are described by model nonlinear equations, and sound beams and pulses are concentrated solutions of these equations. The general issues of the theory of concentrated nonlinear waves are discussed in [3, 4]. Nonlinear acoustics is studied in [5, 6], and the mathematical methods used to study hydrodynamics problems are described in detail in [7].

The main features of nonlinear wave processes are the violation of the superposition principle and the dependence of the properties of the medium on the propagating perturbation. Elastic waves can interact with nonacoustic perturbations, resulting in the formation of coupled waves. For example, nonlinear interaction between acoustic and internal gravity waves occurs in the atmosphere [8]. A change in the properties of the medium due to wave propagation can cause another nonlinear effect — the self-localization of powerful acoustic gravity pulses in the ionosphere [9].

Stratified media and waveguides are of special interest in nonlinear acoustics since they allow directional propagation of acoustic waves. Leble investigated [10] various physical processes occurring in stratified media. Finite-amplitude sound waves in waveguides of nonrectangular section were studied by Keller and Millman [11] and those in cylindrical waveguides by Nozaki and Taniuti [12]. In addition, Zabolotskaya and Shvartsburg analysed the stability of nonlinear pulses in waveguides [13]. In [10–13], the properties of the medium were assumed to be independent on the coordinates of the wave.

Problems of nonlinear acoustics are usually concerned with waves in a medium with weak dispersion. Therefore, accumulation of nonlinear distortions can lead to an increase in the front steepness and a significant change in the characteristics of the wave process. This, however, is not always the case. We examine the Cauchy problem

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for the simple model equation of nonlinear wave theory $\partial u/\partial t + c(u)\,(\partial u/\partial x) = 0$, $u(x,0) = f(x)$. If the function $F(x) = c(f(x))$ increases monotonically, discontinuities are not formed [2]. The evolution of the amplitude and shape of the sound beam due to strong nonlinearity, damping, and diffraction is described by the Khokhlov–Zabolotskaya–Kuznetsov equation [6], and if the beam is homogeneous in the transverse direction, it is described by the Burgers equation. The joint action of nonlinearity and dispersion can result in the establishment of a steady-state mode of perturbation propagation. In the present paper, we study the case of weak nonlinearity, where envelope solitons can form during propagation of amplitude-modulated high-frequency fluctuations.

1. Nonlinear Properties of the Medium. Derivation of the Nonlinear Wave Equation. We consider the propagation of acoustic pulses in a continuously stratified medium, for example, in the atmosphere, in which the equilibrium distribution of the density (and sound velocity) depends on height. Special channels of propagation of infrasonic pulses also arise in auroral zones [14]. We study a gradient layer, i.e., a layer in which the localization of the wave field occurs due to the transverse distribution of the sound velocity. The form of this distribution is established below.

We assume that the axial line of the layer is specified by a certain formula and is characterized by a known curvature $\tilde{\varkappa}(s)$. In this case, it is natural to use the coordinates s (the length of the curve reckoned from a certain specified point) and n (the distance along the normal to the curve). The propagation of acoustic waves is described by the Euler equations

$$\begin{gathered}\frac{\partial\rho}{\partial t} + \frac{\tilde{\varkappa}}{1+n\tilde{\varkappa}}\,\rho v_n + \frac{\partial}{\partial n}\,(\rho v_n) + \frac{1}{1+n\tilde{\varkappa}}\,\frac{\partial}{\partial s}\,(\rho v_s) = 0,\\ \rho\Big(\frac{\partial v_s}{\partial t} + \frac{1}{1+n\tilde{\varkappa}}\,v_s\frac{\partial v_s}{\partial s} + v_n\,\frac{\partial v_s}{\partial n}\Big) + \frac{1}{1+n\tilde{\varkappa}}\,\frac{\partial p}{\partial s} = 0,\\ \rho\Big(\frac{\partial v_n}{\partial t} + \frac{1}{1+n\tilde{\varkappa}}\,v_s\,\frac{\partial v_n}{\partial s} + v_n\,\frac{\partial v_n}{\partial n}\Big) + \frac{\partial p}{\partial n} = 0.\end{gathered} \tag{1}$$

Here v_s and v_n are the longitudinal and normal components of the velocity field, ρ is the density, and p is the pressure. The deviation of the density from the equilibrium value is expressed in terms of the pressure deviation by the equation of state

$$\rho = \rho_0 + (p-p_0)/c_0^2 + \alpha_2(p-p_0)^2/2 + \alpha_3(p-p_0)^3/6 + O((p-p_0)^4), \tag{2}$$

where $\alpha_2 = (\partial^2\rho/\partial p^2)_0$ and $\alpha_3 = (\partial^3\rho/\partial p^3)_0$.

The wave equation can be derived from the basic system (1) by linearization [15]. In the nonlinear case where the sound wave changes the properties of the medium, the Euler equations can also be simplified.

In [11–13], pulse propagation in acoustic waveguides is described by directly using the Euler or Navier–Stokes equations. At the same time, of interest is a model equation which is a generalization of the linear wave equation [15] and provides an accurate determination of the distinguishing features of wave motion.

We examine the self-action of a plane wave

$$p = p_0 + P_*\,\mathrm{e}^{i(ks-\omega t)}, \qquad v_s = V\,\mathrm{e}^{i(ks-\omega t)}, \qquad v_n = 0 \tag{3}$$

in a plane–parallel layer $[\tilde{\varkappa}(s) = 0)]$ ignoring the higher harmonics. Using formulas (2), we eliminate the unknown function ρ from system (1) and then substitute expressions (3) into Eq. (1). After differentiating the first equation in (1) with respect to time and the second with respect to the coordinate s, we subtract one from the other. The nonlinear terms include terms containing an exponential factor in the form of (3); exactly these terms describe the self-action of the wave. Transforming them using the results of linear theory, we arrive at the following conclusion: the effect of nonlinearity on the examined process manifests itself in the fact that the perturbation propagation velocity depends on the wave amplitude or, in other words, the pressure obeys the nonlinear wave equation

$$\frac{\partial^2 p}{\partial s^2} - \frac{1+\lambda|P_*|^2}{c_0^2}\,\frac{\partial^2 p}{\partial t^2} = 0, \tag{4}$$

where

$$\lambda = \alpha_3 c_0^2/2 - \alpha_2/\rho_0 - 1/(\rho_0^2 c_0^4).$$

Equation (4) extends the wave equation of linear acoustics to the case of finite-amplitude waves.

For $\lambda > 0$, the medium is focusing, and for $\lambda < 0$, it is defocusing. In the case of ideally adiabatic sound propagation,

$$\lambda = (2 - 1/\gamma - 3/\gamma^2)/(2p_0^2),$$

in order that the medium be focusing, the adiabatic exponent γ should be larger than 3/2 [16–18]. As is noted in [9, 13], the condition $\lambda > 0$ is rarely satisfied in gas acoustics. This distinguishes the acoustic problem from the problem of electromagnetic pulse propagation in a medium with nonlinear polarizability [19], for which an equation similar to (4) is derived from the Maxwell equations but in which the Kerr coefficient λ is positive for most materials.

For gases, the defocusing action of the medium due to the equations of hydrodynamics can be compensated for only by a strong pressure dependence of sound velocity in a linear approximation $[\alpha_2 = -(2/c_0^3)\,\partial c_0/\partial p]$. In the acoustics of liquids, the empirical Tait equation of state [6], similar to the Poisson equation, is used. In this equation, $\gamma > 3/2$; for example, for water $\gamma = 6.1$. This implies that most liquids are acoustically focusing media.

For the plane wave (3) considered in the derivation of (4), $|P_*|^2 = \langle (p + \bar{p})^2 \rangle/2$, where angular brackets denote averaging over the wave period. Then, Eq. (4) can be extended to the case of perturbations of arbitrary form and to the case of a large number of spatial coordinates ($\partial^2/\partial s^2$ is replaced by the Laplace operator Δ). Thus, investigation of finite-amplitude pulse propagation reduces the basic system (1) to a nonlinear wave equation in which the wave propagation velocity is related to the average squared pressure perturbation for the period according to Eq. (4).

2. Formulation of the Problem and Representation of the Solution. The objective of the present work is to study the evolution of a short pulse propagating in a gradient layer. By analogy to electromagnetic processes, we introduce the function $w^{1/2}(s, n, p) = c_0/c$ ($c_0 = \text{const}$ and c is the local sound velocity), which will be called the acoustic refractive index of the medium. We assume that the examined waveguide layer is strongly inhomogeneous along the transverse coordinate and is weakly inhomogeneous along the longitudinal coordinate. The pulse propagation process is considered weakly nonlinear, and the perturbation amplitude is therefore treated as a quantity of the order of the small parameter ε. The weak longitudinal inhomogeneity is characterized by the parameter ε^2, i.e., w depends on the slow variable $\sigma = \varepsilon^2 s$. From the reasoning given in Sec. 1, it follows that

$$w = \beta^2(n, \sigma) + \alpha(n, \sigma)\langle p^2 \rangle/2 + i\varepsilon^3\Gamma(\sigma). \tag{5}$$

The positive imaginary part of the refractive index formally takes into account energy dissipation in the medium, or, in other words, $\varepsilon^3\Gamma(\sigma)$ is the absorption coefficient. We also assume that the axial line of the waveguide is weakly curved, so that its curvature is $\varepsilon^2\varkappa(\sigma)$. Then, the basic system of Euler equations (1) reduces to the equation for the function $p(s, n, t)$ — the pressure deviation from the equilibrium value during acoustic pulse propagation:

$$\frac{\partial^2 p}{\partial n^2} + \frac{\varepsilon^2\varkappa(\sigma)}{1 + \varepsilon^2\varkappa(\sigma)n}\,\frac{\partial p}{\partial n} + \frac{1}{(1 + \varepsilon^2\varkappa(\sigma)n)^2}\,\frac{\partial^2 p}{\partial s^2} + \frac{\partial}{\partial s}\,\frac{1}{1 + \varepsilon^2\varkappa(\sigma)n}\,\frac{1}{1 + \varepsilon^2\varkappa(\sigma)n}\,\frac{\partial p}{\partial s}$$
$$- \Big(\frac{1}{2}\,\alpha(n, \sigma)\langle p^2 \rangle + \beta^2(n, \sigma) + i\varepsilon^3\Gamma(\sigma)\Big)\frac{\partial^2 p}{\partial t^2} = 0. \tag{6}$$

Since the wave field should be concentrated near the axis of the layer, Eq. (6) needs to be supplemented by the asymptotic boundary condition

$$\lim_{n \to \pm\infty} p = 0. \tag{7}$$

Below, we show that this requirement can be satisfied due to the dependence of the linear part of the function w on the coordinate n, i.e., due to the strong inhomogeneity of the layer in the transverse direction.

In the present paper, the acoustic pulse propagation is consistently split into a faster simple process (the evolution of the fast phase) and a slower complex process (the evolution of the envelope), and, in addition, an asymptotic description of both processes is given. It turns out that the evolution of the envelope is characterized by two scales which take into account the evolution of the envelope phase and the slow variation of its amplitude.

Let us choose a representation for the solution of Eq. (6). We study the weakly nonlinear case; therefore, the wave amplitude is considered to be of the order of ε. In addition, we assume that, in the higher order, the solution is one-mode and that the other modes and higher harmonics have amplitudes of higher orders of smallness.

The form of the solution should take into account the characteristics of the wave traveling along the waveguide axis; therefore, in (6) it is natural to introduce the phase variables

$$\theta = Q(\sigma)/\varepsilon - \varepsilon t, \qquad \theta_m^{(l)} = Q_m^{(l)}(\sigma)/\varepsilon - \varepsilon t,$$

$$m = 1, 2, \ldots, \qquad l = 0, 1, \ldots, K-1,$$

where $Q(\sigma)$ and $Q_m^{(l)}(\sigma)$ are real functions determined by solving the problem; K is the number of modes propagating without damping; the subscript m is the harmonic number, and the superscript l is the mode number. The phase θ is one of the functions $\theta_1^{(l)}$, i.e., $\theta = \theta_1^{(k)}$ (k is a fixed number).

Investigation of the higher harmonics and other modes is beyond the scope of the present work; nevertheless, we write the required function so as to take into account the possibility of these effects. We seek the solution of problem (6), (7) in the form

$$p = \varepsilon P(n, \theta, \sigma, \varepsilon) \exp[i(\varepsilon^{-1}\theta + \varepsilon^{-2}Q_1(\sigma))] + \varepsilon \bar{P}(n, \theta, \sigma, \varepsilon) \exp[-i(\varepsilon^{-1}\theta + \varepsilon^{-2}Q_1(\sigma))]$$

$$+ \varepsilon^2 \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \sum_{\substack{l=0 \\ l \neq k \text{ at } m = \pm 1}}^{K-1} p_m^{(l)}(n, \theta_m^{(l)}, \sigma, \varepsilon) \exp[im(\varepsilon^{-1}\theta_m^{(l)} + \varepsilon^{-2}Q_{1m}^{(l)}(\sigma))], \tag{8}$$

where for the function p is real if the following relations are satisfied: $p_{-m}^{(l)}(n, \theta_{-m}^{(l)}, \sigma, \varepsilon) = \bar{p}_m^{(l)}(n, \theta_m^{(l)}, \sigma, \varepsilon)$ and $\theta_{-m}^{(l)} = \theta_m^{(l)}$. Here $Q_{1(-m)}^{(l)}(\sigma) = Q_{1m}^{(l)}(\sigma)$ are real functions. In (8), the complex amplitudes are expanded in asymptotic series in powers of ε:

$$P = \sum_{j=0}^{\infty} \varepsilon^j P_j(n, \theta, \sigma), \qquad p_m^{(l)} = \sum_{j=0}^{\infty} \varepsilon^j p_{mj}^{(l)}(n, \theta_m^{(l)}, \sigma); \tag{9}$$

accounting for (7) requires that, in the vicinity of the waveguide axis, the following relations be satisfied: $P \underset{n\to\pm\infty}{\to} 0$ and $p_m^{(l)} \underset{n\to\pm\infty}{\to} 0$.

In the propagation process, the mode k is the dominant one. Its characteristics are determined in solving the problem.

3. Phase Functions and Transverse Distribution of the Wave Field. Let us calculate $\langle p^2 \rangle$. The expression for p^2 needs to be integrated over the interval $[\theta, \theta + 2\pi\varepsilon]$ and the result needs to be divided into the interval length. Formula (5) becomes

$$w = \beta^2(n, \sigma) + \varepsilon^2 \alpha(n, \sigma)|P|^2 + \varepsilon^3 \pi \frac{\partial}{\partial \theta}|P|^2 + i\varepsilon^3 \Gamma(\sigma) + O(\varepsilon^4). \tag{10}$$

We substitute the main mode of representation (8) into Eq. (6) written using (10) and (9). Next, we set the terms at identical powers of ε equal to zero and introduce the following notation: $Q_1'(\sigma) = q_1(\sigma)$, $Q'(\sigma) = q(\sigma)$, and $r(\sigma) = q(\sigma) + q_1(\sigma)$.

The complex amplitude of the pulse $P(n, \theta, \sigma, \varepsilon)$ in the higher order in ε is determined by the Sturm–Liouville problem for the variable n with the parameters σ and θ:

$$LP_0 \equiv \frac{\partial^2 P_0}{\partial n^2} + (\beta^2(n, \sigma) - r^2(\sigma))P_0 = 0, \qquad \lim_{n\to\pm\infty} P_0 = 0. \tag{11}$$

Here $r^2(\sigma)$ is an eigenvalue and $P_0(n, \theta, \sigma)$ is an eigenfunction. If $\beta^2(n, \sigma)$ is continuous for all n and $\lim_{n\to\pm\infty} \beta^2(n, \sigma) = -\infty$, there exists an infinite set of eigenvalues $r_l^2(\sigma)$, all of which are simple and can be written as the sequence $r_0^2(\sigma) > r_1^2(\sigma) > r_2^2(\sigma) > \ldots > r_l^2(\sigma) \to -\infty$ [20]. Each eigenfunction has l zeroes, and there exist integrals

$$\int_{-\infty}^{\infty} |P_{0l}|^2\, dn, \qquad \int_{-\infty}^{\infty} \left(\frac{\partial P_{0l}}{\partial n}\right)^2 dn, \qquad \int_{-\infty}^{\infty} \beta^2(n, \sigma)|P_{0l}|^2\, dn.$$

For some values of $\beta^2(n, \sigma)$ in problem (11), there are K positive eigenvalues that correspond to modes that do not damp in σ. The dominant mode in (8) is taken from these modes and is determined by the eigenvalue and the eigenfunction with number k from problem (11).

The form of the operator L allows the required function P_0 to be written as

$$P_0(n, \theta, \sigma) = V(n, \sigma)F(\theta, \sigma),$$

where $V(n,\sigma)$ is a real function normalized by the condition

$$\int_{-\infty}^{\infty} V^2(n,\sigma)\,dn = 1.$$

The function P_0 is a transverse distribution of the field which varies slowly along the waveguide. The complex function $F(\theta,\sigma)$ describes the envelope of the dominant mode and, hence, as a first approximation, the envelope of the entire pulse. We will call this function the pulse envelope.

The next term of series (9) is determined from the inhomogeneous problem

$$LP_1 = 2i(\beta^2(n,\sigma) - q(\sigma)r(\sigma))\,\frac{\partial P_0}{\partial \theta}, \qquad \lim_{n\to\pm\infty} P_1 = 0,$$

whose resolvability condition allows one to calculate $q(\sigma)$:

$$q(\sigma) = r(\sigma) + \frac{1}{r(\sigma)} \int_{-\infty}^{\infty} \Big(\frac{\partial V}{\partial n}\Big)^2\,dn. \tag{12}$$

We denote by $W(n,\sigma)$ the solution of the equation

$$LW = (\beta^2(n,\sigma) - q(\sigma)r(\sigma))V(n,\sigma)$$

that vanishes as $n\to\pm\infty$ and write the function P_1 as

$$P_1(n,\theta,\sigma) = 2iW(n,\sigma)\,\frac{\partial F}{\partial\theta} + c(\theta,\sigma)V(n,\sigma),$$

where $c(\theta,\sigma)$ remains undetermined at the present stage.

Thus, Eqs. (11) and (12) imply expressions for the phase functions of the fast and moderate-rate processes and for the transverse distribution of the pulse field.

4. Evolution of the Pulse Envelope. Another consequence of Eq. (6) and conditions (7) is the problem

$$\begin{aligned} LP_2 = {}& 2i(\beta^2(n,\sigma) - q(\sigma)r(\sigma))\,\frac{\partial P_1}{\partial\theta} + (\beta^2(n,\sigma) - q^2(\sigma))\,\frac{\partial^2 P_0}{\partial\theta^2} - \varkappa(\sigma)\,\frac{\partial P_0}{\partial n} - 2ir(\sigma)\,\frac{\partial P_0}{\partial\sigma} \\ & - ir'(\sigma)P_0 - 2n\varkappa(\sigma)r^2(\sigma)P_0 - \alpha(n,\sigma)|P_0|^2P_0 - \varepsilon\Big(\varkappa(\sigma)\,\frac{\partial P_1}{\partial n} - (\beta^2(n,\sigma) - q^2(\sigma))\,\frac{\partial^2 P_1}{\partial\theta^2} \\ & + 2ir(\sigma)\,\frac{\partial P_1}{\partial\sigma} + ir'(\sigma)P_1 + 2n\varkappa(\sigma)r^2(\sigma)P_1 + 2q(\sigma)\,\frac{\partial^2 P_0}{\partial\theta\,\partial\sigma} + q'(\sigma)\frac{\partial P_0}{\partial\theta} \\ & - 4in\varkappa(\sigma)q(\sigma)r(\sigma)\,\frac{\partial P_0}{\partial\theta} - 2i\alpha(n,\sigma)|P_0|^2\,\frac{\partial P_0}{\partial\theta} + \pi\,\frac{\partial}{\partial\theta}\,|P_0|^2P_0 + i\Gamma(\sigma)P_0\Big), \end{aligned}$$

$$\lim_{n\to\pm\infty} P_2 = 0.$$

The solvability condition for this problem leads to the following equation for the envelope $F(\theta,\sigma)$:

$$\begin{aligned} & 2ir(\sigma)\,\frac{\partial F}{\partial\sigma} + g(\sigma)\,\frac{\partial^2 F}{\partial\theta^2} + j(\sigma)F + h(\sigma)|F|^2F \\ & = -\varepsilon\Big(A(\sigma)\,\frac{\partial F}{\partial\theta} + iB(\sigma)\,\frac{\partial^3 F}{\partial\theta^3} + C(\sigma)|F|^2\,\frac{\partial F}{\partial\theta} + D(\sigma)F^2\,\frac{\partial\bar F}{\partial\theta} + i\Gamma(\sigma)F\Big). \end{aligned} \tag{13}$$

This is a perturbed nonlinear Schrödinger equation with the variable coefficients given by the formulas

$$g(\sigma) = 4\int_{-\infty}^{\infty} (\beta^2(n,\sigma) - q(\sigma)r(\sigma))V(n,\sigma)W(n,\sigma)\,dn - \int_{-\infty}^{\infty} (\beta^2(n,\sigma) - q^2(\sigma))V^2(n,\sigma)\,dn,$$

$$j(\sigma) = ir'(\sigma) + 2\varkappa(\sigma)r^2(\sigma)\int_{-\infty}^{\infty} nV^2(n,\sigma)\,dn,$$

$$h(\sigma) = \int\limits_{-\infty}^{\infty} \alpha(n,\sigma) V^4(n,\sigma)\, dn,$$

$$A(\sigma) = q(\sigma)\Big(\frac{q'(\sigma)}{q(\sigma)} - \frac{r'(\sigma)}{r(\sigma)}\Big) - 4r(\sigma) \int\limits_{-\infty}^{\infty} V(n,\sigma)\, \frac{\partial W}{\partial \sigma}\, dn$$

$$+ i\varkappa(\sigma)\Big(2 \int\limits_{-\infty}^{\infty} V(n,\sigma)\, \frac{\partial W}{\partial n}\, dn + 2r^2 \int\limits_{-\infty}^{\infty} nV(n,\sigma)W(n,\sigma)\, dn$$

$$- 4r^2(\sigma) \int\limits_{-\infty}^{\infty} V(n,\sigma)W(n,\sigma)\, dn \int\limits_{-\infty}^{\infty} nV^2(n,\sigma)\, dn\Big),$$

$$B(\sigma) = \frac{g(\sigma)q(\sigma)}{r(\sigma)} - 2g(\sigma) \int\limits_{-\infty}^{\infty} V(n,\sigma)W(n,\sigma)\, dn$$

$$- 2 \int\limits_{-\infty}^{\infty} (\beta^2(n,\sigma) - q^2(\sigma))V(n,\sigma)W(n,\sigma)\, dn,$$

$$C(\sigma) = 2 \int\limits_{-\infty}^{\infty} \alpha(n,\sigma)V^3(n,\sigma)W(n,\sigma)\, dn - 4ih(\sigma) \int\limits_{-\infty}^{\infty} V(n,\sigma)W(n,\sigma)\, dn$$

$$+ \pi \int\limits_{-\infty}^{\infty} V^4(n,\sigma)\, dn + 2ih(\sigma)\Big(\frac{q(\sigma)}{r(\sigma)} - 1\Big),$$

$$D(\sigma) = \int\limits_{-\infty}^{\infty} \alpha(n,\sigma)V^3(n,\sigma)W(n,\sigma)\, dn - 2ih(\sigma) \int\limits_{-\infty}^{\infty} V(n,\sigma)W(n,\sigma)\, dn$$

$$+ \pi \int\limits_{-\infty}^{\infty} V^4(n,\sigma)\, dn + i\,\frac{h(\sigma)q(\sigma)}{r(\sigma)}.$$

It is easy to see that the term $i[\operatorname{Im} j(\sigma) + \varepsilon\Gamma(\sigma)]F$ in (13) characterizes the damping of the pulse. In this case, the energy absorption by the medium, which is determined by the coefficient Γ, also occurs in the case where the properties of the waveguide do not depend on the variable σ whereas the contribution of the quantity $\operatorname{Im} j(\sigma)$ to the damping is due to the weak longitudinal inhomogeneity.

Let us study in more detail the unperturbed nonlinear Schrödinger equation, i.e., Eq. (13) for $\varepsilon = 0$. Of interest are its solutions concentrated in the variable θ because they correspond to the propagation of short pulses in the waveguide layer. Mathematically, the concentration means that the solution of the nonlinear Schrödinger equation is a function such that, as $\theta \to \pm\infty$, it tends to zero, with all derivatives with respect to θ, faster than any power $|\theta|^{-1}$.

As shown in [19], if the coefficients of the nonlinear Schrödinger equation are linked by the relation

$$2g(\sigma)r(\sigma) = \lambda^2 h(\sigma), \qquad \lambda = \text{const}, \tag{14}$$

its concentrated solution is the function

$$F(\theta,\sigma) = \frac{\lambda}{\sqrt{r(\sigma)}}\, \exp\Big[i\Big(\theta + \frac{1}{2}\int\limits_0^{\sigma} \frac{\operatorname{Re} j(\sigma')}{r(\sigma')}\, d\sigma'\Big)\Big] \Big/ \cosh\Big(\theta - \int\limits_0^{\sigma} \frac{g(\sigma')}{r(\sigma')}\, d\sigma'\Big). \tag{15}$$

The expression for p written with the use of the formulas for $V(n,\sigma)$, $F(\theta,\sigma)$, $r(\sigma)$, and $q(\sigma)$ clearly reflects the nature of the examined motion, determined by three scales: the high-frequency filling is modulated by the envelope, whose evolution depends on two scales. In addition, Eq. (15) implies that the pulse propagates in a longitudinally inhomogeneous waveguide at the velocity

$$v(\sigma) = \frac{1}{q(\sigma)}\Big(1 + \varepsilon\,\frac{q(\sigma)r(\sigma)}{g(\sigma)} + O(\varepsilon^2)\Big),$$

which varies slowly during propagation.

In problems of nonlinear acoustics, the case $\alpha < 0$ with the coefficient $h(\sigma) < 0$ is also of interest. Let the following relation be satisfied:

$$2g(\sigma)r(\sigma) = -\lambda^2 h(\sigma), \qquad \lambda = \text{const.}$$

Then,

$$F(\theta,\sigma) = \frac{\lambda}{\sqrt{r(\sigma)}}\tanh\Big(\theta - \int_0^\sigma \frac{g(\sigma')}{r(\sigma')}\,d\sigma'\Big)\exp\Big[i\Big(\theta + \int_0^\sigma \frac{\operatorname{Re} j(\sigma') - 3g(\sigma')}{2r(\sigma')}\,d\sigma'\Big)\Big]. \tag{15a}$$

While propagating, the pulse (15a) extinguishes the high-frequency filling. This pulse (by analogy with the electromagnetic problem) is called a dark soliton or an envelope hole. As noted in [12], this type of process is characteristic of cylindrical acoustic waveguides filled with air.

5. Propagation of a Pulse in a Waveguide with a Parabolic Profile of the Refractive Index. Many formulas from Sec. 1–4 are considerably simplified by assuming that

$$\beta^2(n,\sigma) = \beta_0^2(\sigma) - \beta_2^2(\sigma)n^2/4.$$

For large values of the normal coordinate, $\beta^2(n,\sigma) < 0$, but for finding a solution in the vicinity of the waveguide axis, this approximation is justified.

The eigenvalues and eigenfunctions of the Sturm–Liouville problem (11) are equal, respectively, to

$$r_l^2(\sigma) = \beta_0^2(\sigma) - (l+1/2)\beta_2(\sigma), \qquad V_l(n,\sigma) = (2^2 l!)^{-1/2}(2\pi)^{-1/4}\beta_2^{1/4}(\sigma)D_l(\sqrt{\beta_2(\sigma)}\,n).$$

Here $D_l(x)$ is a parabolic cylinder function. The existence of $K = [\beta_0^2/\beta_2 + 1/2]$ undamped modes ($[x]$ is the integer part of x) is possible. We examine the mode with number $k = 0$ assuming that $\alpha = \alpha(\sigma)$. We have

$$r(\sigma) = \sqrt{\beta_0^2 - \beta_2/2}, \qquad q(\sigma) = (\beta_0^2 - \beta_2/4)/\sqrt{\beta_0^2 - \beta_2/2},$$

$$V(n,\sigma) = (\beta_2/(2\pi))^{1/4}\,\mathrm{e}^{-\beta_2 n^2/4}, \qquad W(n,\sigma) = \beta_2(\sigma)n^2 V(n,\sigma)/8.$$

The coefficients of the nonlinear Schrödinger equation can also be calculated in terms of α, β_0, and β_2. If the relation

$$\alpha(\sigma) = \frac{\sqrt{\pi}}{\lambda^2}\,\frac{\beta_0(\sigma)\sqrt{\beta_2(\sigma)}}{\sqrt{1 - \beta_2(\sigma)/(2\beta_0^2(\sigma))}}$$

is satisfied, the pulse envelope is given by the formula

$$F(\theta,\sigma) = \lambda\,\mathrm{e}^{i\theta} \Big/ \Big\{\Big(\beta_0^2(\sigma) - \frac{1}{2}\beta_2(\sigma)\Big)^{1/4}\cosh\Big[\theta - \frac{1}{4}\int_0^\sigma \frac{\beta_0^2(\sigma')\beta_2(\sigma')}{(\beta_0^2(\sigma') - \beta_2(\sigma')/2)^{3/2}}\,d\sigma'\Big]\Big\}.$$

For $\beta_2 \ll \beta_0^2$, the velocity of the zero mode is close to the pulse velocity in the linear approximation $1/\beta_0(\sigma)$. For large values of β_2, the velocity $v(\sigma) < 1/\beta_0(\sigma)$.

6. Effect of Longitudinal Inhomogeneity on Pulse Concentration, Amplitude, and Width. As noted above, if the coefficients $r(\sigma)$, $g(\sigma)$, and $h(\sigma)$ of the unperturbed nonlinear Schrödinger equation are linked by relation (14), its solution is concentrated in θ for all σ. We examine the case where the coefficients are arbitrary.

We set $\varepsilon = 0$ and further simplify Eq. (13). For this, we represent the envelope F as the product

$$F(\theta,\sigma) = f(\theta,y)\exp\Big(i\int_0^\sigma \frac{j(\sigma')}{2r(\sigma')}\,d\sigma'\Big)$$

by introducing the new variable

$$y = \frac{1}{2} \int\limits_0^\sigma \frac{g(\sigma')}{r(\sigma')}\, d\sigma'.$$

The exponential multiplier explicitly describes the pulse damping due to the inhomogeneity of the waveguide layer and the additional incursion of the envelope phase.

Due to the longitudinal inhomogeneity of the waveguide, the pulse concentrated at the beginning of the path can break up. The problem of the conservation of the pulse concentration reduces to investigating the concentration of the function $f(\theta, y)$, satisfying the nonlinear Schrödinger equation

$$i\,\frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial \theta^2} + H(y)|f|^2 f = 0, \tag{16}$$

where $H(y) = h(y)/(r(y)g(y))$; the variable σ is expressed in terms of y.

The breakup of the pulse is favored by a high power and steepness of the initial pulse $f(\theta, 0) = \Psi(\theta)$ and significant longitudinal inhomogeneity. As shown in [21, 22] if $H(y)$ is a continuous positive function, that exists a guaranteed interval $[0, Y)$ in which the pulse concentration is conserved. The value of Y is determined from the equation

$$6 \int\limits_{-\infty}^{\infty} (|\Psi|^2 + |\Psi'|^2)\, d\theta \int\limits_0^Y H(y)\, dy = 1. \tag{17}$$

Knowing Y and the formula of change of the variables $y(\sigma)$, one can calculate the length of the guaranteed interval for the initial variable σ. Equation (17) implies that the length of the guaranteed interval decreases with increasing power and steepness of the initial pulse.

It should be noted that $[0, Y)$ is the minimum interval in which the pulse concentration is conserved. The solution of Eq. (16) belonging to the class of rapidly decreasing functions can also exist over a wider range. If

$$\int\limits_0^\infty |H'(y)|\, dy < \infty, \tag{18}$$

the pulse concentration is conserved for any rapidly decreasing functions $\Psi(\theta)$ over the entire length of the waveguide layer [21].

We examine Eq. (16) by assuming that the longitudinal inhomogeneity is not only small but is also smoothly varying. In other words,

$$H(y) = 2\mu^2 - \delta b(\delta y), \qquad \mu = \text{const}, \qquad \delta \ll 1,$$

where $b(\xi)$ is a bounded real function that depends on the slow variable $\xi = \delta y$. Then, using the methods described in [3, 4], one can construct a solution of Eq. (16) that is asymptotic in the small parameter δ.

Representing the required function as $f = \Phi\, \mathrm{e}^{i\varphi}$ (Φ, φ are real functions) and taking into account the shape of the soliton solution for $\delta = 0$, we find solution of (16) in the form of series in the powers of δ:

$$\varphi = \theta\varphi_0(\xi) + \delta\varphi_1(X, \xi) + \delta^2\varphi_2(X, \xi) + \dots,$$

$$\Phi = A_0(\xi)/\cosh X + \delta\Phi_1(X, \xi) + \delta^2\Phi_2(X, \xi) + \dots, \qquad X = \theta - x_0(\xi)/\delta - x_1(\xi).$$

Substitution of these series into (16) yields the following conditions in the higher orders in δ

$$x_0'(\xi) = 2, \qquad x_1'(\xi) = x_1 = \text{const}, \qquad A_0(\xi) = 1/\mu, \qquad \varphi_0(\xi) = 1,$$

$$\frac{\partial^2\Phi_1}{\partial X^2} + \Big(\frac{6}{\cosh^2 X} - 1\Big)\Phi_1 = \frac{b(\xi)}{\mu^3 \cosh^3 X}, \qquad \Phi_1 \underset{X\to\pm\infty}{\to} 0,$$

$$\frac{\partial^2\varphi_1}{\partial X^2}\,\frac{1}{\cosh X} - 2\,\frac{\partial\varphi_1}{\partial X}\,\frac{\sinh X}{\cosh^2 X} = -x_1'(\xi)\,\frac{\sinh X}{\cosh^2 X}.$$

The equations for Φ_1 and φ_1 are easily integrated. As a result, the required function f is represented as

$$f(\theta, y) = \frac{1}{\mu} \frac{1}{\cosh\,(\theta - 2y - \delta x_1 y)} \Big(1 + \frac{\delta b(\xi)}{4\mu^2} + \delta d(\xi) \tanh\,(\theta - 2y - \delta x_1 y) + O(\delta^2)\Big) \exp\Big(i\theta + i\delta\,\frac{x_1 X}{2}\Big), \quad (19)$$

where $d(\xi)$ is a solution of a second-order nonlinear equation with the coefficients expressed in terms of the function $b(\xi)$.

Equation (19) implies the following expressions for the amplitude A and the width Δ of the pulse envelope:

$$A = \frac{1}{\mu} \exp\Big(-\frac{1}{2}\int_0^\sigma \frac{\operatorname{Im} j(\sigma')}{r(\sigma')}\, d\sigma'\Big)\Big(1 + \frac{\delta}{4\mu^2}\, b(\delta y) + O(\delta^2)\Big), \qquad \Delta = \Delta_0\, \frac{r(\sigma)}{g(\sigma)}\,(1 + O(\delta)) \quad (20)$$

(Δ_0 is the initial pulse width and μ^{-1} is the initial pulse amplitude). The amplitude and width vary during pulse propagation. If, for example, the ratio g/r increases with increasing σ, Δ decreases. This implies that weak longitudinal inhomogeneity of the waveguide layer can lead to a significant compression of the pulse.

Formulas (20) are valid in the waveguide region $y < Y$. However, if condition (18) is satisfied, these formulas describe the pulse amplitude and width over the entire path.

If the pulse broke up, the gradients of the physical quantities become large and the viscosity of the medium cannot be ignored. In this case, the Euler equations are inapplicable.

Conclusions. The analysis of the propagation of a short pulse in a gradient inhomogeneous waveguide leads to the following conclusions. Many problems of nonlinear acoustics are related to investigation of strong nonlinearity, which is responsible, in particular, for the formation of shock waves. Pulse propagation in nonlinear media with dispersion and dissipation are modeled using the Burgers, Korteweg–de Vries, Khokhlov–Zabolotskaya–Kuznetsov, etc., equations. Dispersion can compensate for the increase in the wave profile steepness due to nonlinearity, resulting in the occurrence of a pulse propagating without shape change. The present study shows that, in the case of weak nonlinearity, there is another effect, namely, the pulse is amplitude-modulated high-frequency sinusoidal oscillations (probably, distorted by the presence of longitudinal inhomogeneity). In contrast to strong nonlinearity, weak nonlinearity does not influence the high-frequency filling, and its effect is only manifested in the formation of solitons of the pulse envelope. In this case, the pulse propagation process is determined by three scales. The amplitude-modulated high-frequency filling evolves the most rapidly. The evolution of the envelope is determined by two processes, which proceed at different velocities: the moderate-velocity evolution of the envelope phase and the slow variation of its amplitude. If the waveguide is inhomogeneous in the longitudinal direction, the envelope is described by the nonlinear Schrödinger equation with variable coefficients.

Equation (13) takes into account the damping of the pulse amplitude along the s coordinate. A similar result was obtained in a study [12] of a homogeneous waveguide with walls using a nonlinear Schrödinger equation with constant coefficients. In [12], however, damping is due not only to dissipation but also to radial flows in the boundary layer, i.e., ultimately, to the presence of the boundary. In the case considered, additional damping is due to the dependence of the properties of the waveguide on the longitudinal coordinate.

It should be noted that recent mathematical studies of the solvability of the Navier–Stokes equations revealed a difference in the nature of the solutions for $\gamma > 3/2$ [23] and $\gamma < 3/2$ [24].

Numerical calculations [12] have shown that, for all modes in air ($\gamma \approx 1.41$), only propagation of envelope holes is possible, which corresponds to a defocusing medium. This result agrees with the above statement that, for ideally adiabatic propagation of sound pulses, $\gamma_0 = 1.5$ is the boundary value of γ_0 which separates focusing and defocusing media. The conclusion that the focusing nature of a medium can be provided by a strong pressure dependence of sound velocity also agrees with the results of [12]. A comparison with [12] shows that investigation of gradient waveguide layers eliminates difficulties arising in accounting for the effect of the boundaries. At the same time, in stratified media, the sound velocity distribution can be responsible for the occurrence of a gradient waveguide in practice. In such waveguides, the propagation of a concentrated acoustic pulse, at least, for finite distances, is possible. The transverse distribution of the wave field and the phase functions of the high-frequency filling and the envelope are found by solving the Sturm–Liouville problem (11), and the pulse envelop can be studied using Eq. (13).

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